THE PARITY OF THETA CHARACTERISTICS IS PRESERVED BY INFINITESIMAL DEFORMATIONS

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Abstract. In this note, given a family of relative dimension one over a smooth curve, we determine the parity of the restriction of a relative theta characteristic to an arbitrary multiple of a fiber in terms of the parity of the restriction to a general fibre.
This result can be regarded as a variant of the well-known theorem on the invariance of the parity of theta characteristics in families.
As a corollary, we obtain that the torsion subsheaf of the first higher direct image sheaf of a relative theta characteristic splits as a direct sum of two isomorphic sheaves.

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Contents

1. Introduction

We work over an algebraically closed field $\mathbb{K}$ of characteristic $\neq 2$.

In [Mu71] Mumford has given an algebraic proof, independent of the theory of theta functions, of the classically known fact that the parity of theta-characteristics is constant in families. We recall below a generalized version of this result, due to Harris (see also [Co89]):

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We are indebted to Barbara Fantechi for explaining us how to use Cohomology and Base Change in this context.

Theorem. ([Ha82, Theorem 1.10.(i)]) *Let Δ be an irreducible variety, let $\pi\colon X \to \Delta$ be a proper flat map with fibers $C_t := \pi^{-1}(t)$ reduced curves, let $\mathcal{L}$ be a line bundle on X and set $\mathcal{L}_t := \mathcal{L}|_{C_t}$.*

If $\mathcal{L}_t^{\otimes 2} \cong \omega_{C_t}$ for all $t \in \Delta$, then the function $t \mapsto h^0(C_t, \mathcal{L}_t)$ is constant modulo 2.

In this note we prove an infinitesimal version of the above result:

Theorem 1.1. *Let Δ be a smooth connected curve and let $\pi\colon X \to \Delta$ a projective flat morphism whose fibers $C_t := \pi^{-1}(t)$ are reduced connected curves and let $\mathcal{L}$ be a line bundle on X.*

If $\mathcal{L}_t^{\otimes 2} \cong \omega_{C_t}$ for all $t \in \Delta$, then $h^0(kC_t, \mathcal{L}|_{kC_t}) = kh^0(C_t, \mathcal{L}_t)$ modulo 2 for all $k \in \mathbb{N}_{>0}$ and for all $t \in \Delta$.

Fixed $t \in \Delta$, the even numbers $kh^0(C_t, \mathcal{L}_t) - h^0(kC_t, \mathcal{L}|_{kC_t})$ form a non-decreasing sequence, indexed by k.

The last sentence may be seen as an infinitesimal version of semicontinuity. Combining the two previous theorems, we obtain:

Corollary 1.2. *In the assumptions of Theorem 1.1, there is a coherent sheaf $\mathcal{T}$ on Δ such that the torsion subsheaf of $R^1\pi_*\mathcal{L}$ is isomorphic to $\mathcal{T} \oplus \mathcal{T}$.*

The proofs are given in the next section.

Our interest in this question arose in studying surfaces of general type with canonical map of odd degree (cf. [MLPP26]). The fact that the parity of theta characteristics is constant in families is crucial throughout our analysis of such surfaces but it does not suffice to deal with some of the possible cases, that we finally managed to rule out by means of Corollary 1.2.

We think however that Theorem 1.1 and Corollary 1.2 are of independent interest. Actually, as suggested by J. Kollár, whom we thank warmly, it is natural to ask whether Theorem 1.1 can be generalized to the case where B is any local artinian $\mathbb{K}$-algebra. Namely, given a flat family of curves $\mathcal{X} \to \operatorname{Spec} B$ with closed fiber X_0 and a relative theta characteristic $\mathcal{L}$ on $\mathcal{X}$, is it true that the number $\dim_{\mathbb{K}} B \cdot h^0(\mathcal{L}|_{X_0}) - h^0(\mathcal{X}, \mathcal{L})$ is even?

We do not know the answer to this question; we just observe that our method of proof does not extend to the general situation (cf. Example 2.2).

2. Proofs

2.1. **Preliminary results.** The proof of Theorem 1.1 uses some auxiliary results that we now explain.

Let Δ be a smooth connected curve, fix $\bar{t} \in \Delta$, set $A := \mathcal{O}_{\Delta,\bar{t}}$, $B_k := A/s^kA$, where $s \in A$ a local parameter.

Lemma 2.1. *Let $q > 0$ be an integer and let $\psi\colon A^q \to A^q$ be an A-linear map given by a skew-symmetric matrix M; for $k \in \mathbb{N}_{>0}$ let $\psi_k\colon B_k^q \to B_k^q$ be the map induced by M and let r_k be the dimension of $\operatorname{Im}\psi_k$ as a $\mathbb{K}$-vector space.*

Then $\{r_k\}$ *is a non-decreasing sequence of even integers.*

Proof. Writing M as

$$M = \sum_{j=0}^{k-1} s^j M_j \qquad \mod s^k$$

we get skew-symmetric matrices M_j with coefficients in $\mathbb{K}$. Denote by r_k the dimension as a $\mathbb{K}$-vector space of the image of ψ_k. Let $c_1, \dots, c_q$ be the standard basis of B_k^q as a B_k-module. Then

$$c_1, \dots, c_q, sc_1, \dots, sc_q, s^2c_1, \dots, s^{k-1}c_q \tag{2.1}$$

is a basis of B_k^q as a $\mathbb{K}$-vector space. We now use this basis to associate a matrix with the operator ψ_k, but we order it differently when using it as a basis for the domain or as a basis for the codomain. Precisely we order it as in (2.1) as a basis of the domain and as

$$s^{k-1}c_1, \dots, s^{k-1}c_q, s^{k-2}c_1, \dots, s^{k-2}c_q, s^{k-3}c_1, \dots, c_q$$

as a basis of the codomain. Then a straightforward computation shows that the matrix associated to ψ_k with respect to this choice of the bases is the block matrix

$$N_k := \begin{pmatrix} M_{k-1} & M_{k-2} & M_{k-3} & \cdots & M_0 \\ M_{k-2} & M_{k-3} & M_{k-4} & \cdots & 0 \\ M_{k-3} & M_{k-4} & M_{k-5} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ M_0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

As all the matrices M_j are skewsymmetric, N_k is skewsymmetric as well, and therefore r_k, which equals the rank of N_k, is even.

Finally, we note that N_k is a submatrix of N_{k+1}, and therefore the sequence of even numbers $\{r_k\}_{k\in\mathbb{N}}$ is non-decreasing. □

Lemma 2.1 is key for the proof of Theorem 1.1. The following example shows that it does not hold in general for a local $\mathbb{K}$-algebra A and an artinian quotient $B = A/I$.

Example 2.2. Set $A = \mathbb{K}[x, y]$, $I = (x, y)^2$ and $B := A/I$. Consider the skewsymmetric matrix

$$M := \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ -x & -y & 0 \end{pmatrix}$$

and let $\overline{\psi}\colon B^3 \to B^3$ be the map induced by M. The image of $\overline{\psi}$ is spanned as a $\mathbb{K}$-vector space by the independent vectors:

$${}^t(0, 0, x), \quad {}^t(0, 0, y), \quad {}^t(x, y, 0)$$

and therefore it has odd dimension.

Let V be a free A-module of rank $2r$ with a symmetric bilinear form $Q\colon V\times V\to A$ whose reduction modulo s, $\overline{Q}\colon \overline{V}\times\overline{V}\to\mathbb{K}$, is non degenerate. Assume that $W_1, W_2\subset V$ are free rank r submodules such that V/W_i is free for $i=1,2$. In particular, the map $W_i\otimes_A B_k\to V\otimes_A B_k$ is injective for $i=1,2$ for all k.

Set now, for all $k\geq 1$

$$q_k := \dim_{\mathbb{K}}\left((W_1\otimes_A B_k)\cap(W_2\otimes_A B_k)\rangle\right)$$

Then we get the following infinitesimal version of the permanence of the parity of the dimension of the intersection of two maximal isotropic subspaces:

Lemma 2.3. *In the above set-up, if W_1 and W_2 are totally isotropic for Q, then the sequence $\{kq_1-q_k\}_{k\in\mathbb{N}}$ is a non-decreasing sequence of even numbers.*

Proof. Given an A-module N we write $\overline{N}:=N/sN$ and for $z\in N$ we denote by $\bar{z}\in\overline{N}$ its image.

We start by showing that any basis $e_1,\dots e_r$ of W_1 as A-module can be completed to a basis $e_1,\dots e_r, f_1,\dots f_r$ of V such that $Q(f_i,f_j)=0$ and $Q(f_i,e_j)=\delta_{ij}$ for all $1\leq i,j\leq r$.

The natural map $V\to W_1^\vee$ induced by Q is surjective, since W_1 is a direct summand of V and $\overline{Q}$ is non degenerate. So we may find $w_1,\dots w_r\in V$ such that $Q(w_i,e_j)=\delta_{ij}$ for all $1\leq i,j\leq r$ and set $f_i:=w_i-\frac{1}{2}\sum_{j=1}^r Q(w_i,w_j)e_j$.

We set for sake of simplicity $q:=q_1$. By definition of q_k, $\dim_{\mathbb{K}}\left(\overline{W}_1\cap\overline{W}_2\right)=q$. We choose a basis $e_1,\dots e_r$ of W_1 such that $\bar{e}_1,\dots\bar{e}_q$ is a basis of $\overline{W}_1\cap\overline{W}_2$ and complete it to a basis $e_1,\dots e_r,f_1,\dots f_r$ of V as above.

One can pick a basis $v_1,\dots v_r$ of W_2 such that $\bar{v}_i=\bar{e}_i$ for $i=1,\dots q$ and such that $\bar{v}_i=\bar{f}_i+\sum_{j=q+1}^r a_{ij}\bar{e}_j$ for $i=q+1,\dots r$ and some scalars $a_{ij}\in\mathbb{K}$. Since $\overline{W}_2$ is a totally isotropic subspace, the matrix (a_{ij}) is skewsymmetric. So, replacing f_i by $f_i+\sum_{j=q+1}^r a_{ij}e_j$ for $i=q+1,\dots r$, we may assume in addition that $\overline{v}_i=\bar{f}_i$ for $i=q+1,\dots r$.

Denote by $U_1\subset V$ the span of $e_1,\dots e_q,f_{q+1},\dots f_r$ and by U_2 the span of $f_1,\dots f_q,e_{q+1},\dots e_r$, so that $V=U_1\oplus U_2$ is a decomposition as the sum of totally isotropic subspaces. As $\overline{W}_2=\overline{U}_1$, by Nakayama's Lemma the projection $V\to U_1$ with kernel U_2 restricts to a surjective map, hence an isomorphism, $W_2\to U_1$. So we may write

$$v_i=e_i+sz_i \text{ for } i=1,\dots,q \qquad v_i=f_i+sz_i \text{ for } i=q+1,\dots,r$$

where

$$z_i=\sum_{j=q+1}^r\lambda_{ij}e_j+\sum_{j=1}^q\mu_{ij}f_j \tag{2.2}$$

Then $(W_1 \otimes_A B_k) + (W_2 \otimes_A B_k)$ is generated, as a $\mathbb{K}$-vector space, by the classes modulo s^k of:

$$\begin{gathered} e_1, \dots e_r, \dots, s^{k-1}e_1, \dots s^{k-1}e_r, \\ v_{q+1}, \dots v_r, \dots, s^{k-1}v_{q+1}, \dots s^{k-1}v_r \\ s\sum_{j=1}^{q}\mu_{1j}f_j, \dots, s\sum_{j=1}^{q}\mu_{qj}f_j, \dots, s^{k-1}\sum_{j=1}^{q}\mu_{1j}f_j, \dots, s^{k-1}\sum_{j=1}^{q}\mu_{qj}f_j. \end{gathered} \tag{2.3}$$

It is easy to see that for every vanishing linear combination with coefficients in $\mathbb{K}$ of the classes in (2.3) the coefficients of the classes in the first two rows are trivial, and therefore

$$\dim_{\mathbb{C}}\left((W_1 \otimes_A B_k) + (W_2 \otimes_A B_k)\right) = k(2r-q) + r_k$$

where r_k is the dimension of the complex vector subspace of $V \otimes_A B_k$ generated by the classes in the last row of (2.3).

By the Grassman formula

$$q_k = \dim_{\mathbb{C}}\left((W_1 \otimes_A B_k) \cap (W_2 \otimes_A B_k)\right) = kq - r_k.$$

Consider the $q \times q$ matrix $M = (s\mu_{ij})_{i=1,\dots,q}^{j=1,\dots,q}$ with entries in A. Since W_2 is totally isotropic, for $1 \le i,j \le q$ one has $0 = Q(v_i, v_j) = s(\mu_{ij} + \mu_{ji}) + s^2 Q(z_i, z_j)$. By (2.2) $Q(z_i, z_j) = 0$ and therefore $\mu_{ij} + \mu_{ji} = 0$. So M is skewsymmetric and r_k is a non-decreasing sequence of even numbers by Lemma 2.1. □

We recall also the following well known fact:

Lemma 2.4. *Let $\Delta := \operatorname{Spec} R$ be a smooth affine curve, let $\pi\colon X \to \Delta$ be a proper morphism with 1-dimensional fibers and let F be a coherent sheaf on X flat over Δ. Then there exists a two term complex $M^\bullet : 0 \to M^0 \to M^1 \to 0$ of finitely generated locally free modules and an isomorphism of functors*

$$H^p(X \times_\Delta \operatorname{Spec} B, F \otimes_R B) \cong H^p(M^\bullet \otimes_R B), \quad p \ge 0$$

on the category of R-algebras B.

Proof. By the Theorem in §5 of Chapter II of [Mu70] there is a finite complex $K^\bullet : 0 \to K^0 \to \cdots \to K^n \to 0$ of finitely generated projective R-modules and an isomorphism of functors

$$H^p(X \times_\Delta \operatorname{Spec} B, F \otimes_R B) \cong H^p(K^\bullet \otimes_R B), \quad p \ge 0$$

on the category of R-algebras. Since Δ is a smooth curve the modules $K^\bullet$ and all their submodules are locally free. Set $M^0 := K^0$ and $M^1 := \ker(K^1 \to K^2)$. The complex $M^\bullet := 0 \to M^0 \to M^1 \to 0$ has a natural map to $K^\bullet$ which is a quasi-isomorphism, since the fibers of π are 1-dimensional. We conclude by applying Lemma 2 in §5 of Chapter II of [Mu70] to the map of complexes of flat modules $M^\bullet \to K^\bullet$. □

Corollary 2.5. *In the assumptions of Lemma 2.4, if B is an R-algebra then:*

(i) $H^1(X \times_\Delta B, F \otimes_R B) \cong H^1(X, F) \otimes_R B$
(ii) *if* $H^1(X, F) = 0$, *then* $H^0(X \times_\Delta B, F \otimes_R B) \cong H^0(X, F) \otimes_R B$

Proof. Let $M^\bullet := 0 \to M^0 \to M^1 \to 0$ be the complex given by Lemma 2.4. There is an exact sequence

$$0 \to H^0(X, F) \to M^0 \to M^1 \to H^1(X, F) \to 0 \tag{2.4}$$

and $H^i(X \times_\Delta B, F \otimes_R B)$ is the i-th cohomology of $M^\bullet \otimes_R B$. So (i) is a consequence of the fact that tensor product is right exact.

If $H^1(X, F) = 0$, then (2.4) gives a short exact sequence

$$0 \to H^0(X, F) \to M^0 \to M^1 \to 0 \tag{2.5}$$

Since M^1 is locally free, and flatness is a local property (see [TSP] Lemma 00HT) then M^1 is flat and therefore, by Lemma 00HL of [TSP], (2.5) stays exact after tensoring with B. So $H^0(X, F) \otimes_R B \cong H^0(M^\bullet \otimes_R B)$, proving (ii). □

2.2. **Proof of Thm. 1.1.** The proof follows the same lines as the proofs of [Mu71, §1] and [Ha82, Theorem 1.10.(i)].

The statement is local, hence we work near a fixed point $\bar{t} \in \Delta$ and we denote by C the fiber $C_{\bar{t}}$ and by L the line bundle $\mathcal{L}_{\bar{t}} = \mathcal{L}|_{C_{\bar{t}}}$. We pick $p_1, \dots p_N$ smooth points of C such that $h^0(C, L(-D)) = h^1(C, L(D)) = 0$, where $D = \sum p_i$, and consider the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L(-D) & \longrightarrow & L & \longrightarrow & L/L(-D) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L(-D) & \longrightarrow & L(D) & \longrightarrow & L(D)/L(-D) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L(D)/L & = & L(D)/L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2.6}$$

Taking cohomology in (2.6) one gets another exact diagram:

(2.7)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(L) & \longrightarrow & H^0(L/L(-D)) & \longrightarrow & H^1(L(-D)) & & \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H^0(L(D)) & \longrightarrow & H^0(L(D)/L(-D)) & \longrightarrow & H^1(L(-D)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & H^0(L(D)/L) & = & H^0(L(D)/L) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Set $\overline{V} := H^0(L(D)/L(-D))$, $\overline{W}_1 := H^0(L/L(-D))$, $\overline{W}_2 := H^0(L(D))$. Arguing as in [Mu71, §1] and in the proof of [Ha82, Theorem 1.10.(i)] one shows that $\dim \overline{V} = 2N$ and, using the isomorphism $2L \cong \omega_C$ constructs a non-degenerate bilinear form $\overline{Q}\colon \overline{V} \times \overline{V} \to \mathbb{K}$ such that $\overline{W}_1$ and $\overline{W}_2$ are maximal isotropic subspaces. Chasing through diagram (2.7) it is easy to check that $H^0(L)$ can be identified with $\overline{W}_1 \cap \overline{W}_2$.

The next step, still following [Mu71, §1] and the proof of [Ha82, Theorem 1.10.(i)], consists in giving a relative version of this construction. Étale locally we may assume that $p_1, \dots p_N$ are cut out on C by disjoint sections $\sigma_1, \dots \sigma_N$ of π contained in the smooth locus of X. Write $\mathcal{D} := \sigma_1 + \cdots + \sigma_N$; up to shrinking Δ we may assume $\pi_*\mathcal{L}(-\mathcal{D})) = R^1\pi_*(\mathcal{L}(\mathcal{D})) = 0$. Then we have the following exact diagram, the relative version of (2.7):

(2.8)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_*\mathcal{L} & \longrightarrow & \pi_*(\mathcal{L}/\mathcal{L}(-\mathcal{D})) & \longrightarrow & R^1\pi_*(\mathcal{L}(-\mathcal{D})) & & \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \pi_*(\mathcal{L}(\mathcal{D})) & \longrightarrow & \pi_*(\mathcal{L}(\mathcal{D})/\mathcal{L}(-\mathcal{D})) & \longrightarrow & R^1\pi_*(\mathcal{L}(-\mathcal{D})) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \pi_*(\mathcal{L}(\mathcal{D})/\mathcal{L}) & = & \pi_*(\mathcal{L}(\mathcal{D})/\mathcal{L}) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

We observe that, possibly up to shrinking Δ further, we may assume:

- $R^1\pi_*\mathcal{L}(-\mathcal{D})$ is free, since it has constant rank, hence the middle row of diagram (2.8) is split

- $\pi_*(\mathcal{L}(\mathcal{D})/\mathcal{L})$ is also free, since $\mathcal{D} \to \Delta$ is a finite flat map, hence the middle column of diagram (2.8) is also split
- the sheaves $\mathcal{V} := \pi_*(\mathcal{L}(\mathcal{D})/\mathcal{L}(-\mathcal{D}))$, $\mathcal{W}_1 := \pi_*(\mathcal{L}/\mathcal{L}(-\mathcal{D}))$ and $\mathcal{W}_2 := \pi_*(\mathcal{L}(\mathcal{D}))$ are free

Set $A := \mathcal{O}_{\Delta,\bar{t}}$, let $s \in A$ be a local parameter and let $B_k := A/s^k A$, where $0 < k \in \mathbb{N}$. Tensoring diagram (2.8) with B_k gives:

(2.9)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathcal{W}_1 \otimes B_k) \cap (\mathcal{W}_2 \otimes B_k) & \longrightarrow & \mathcal{W}_1 \otimes B_k & \longrightarrow & R^1\pi_*(\mathcal{L}(-\mathcal{D})) \otimes B_k & & \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{W}_2 \otimes B_k & \longrightarrow & \mathcal{V} \otimes B_k & \longrightarrow & R^1\pi_*(\mathcal{L}(-\mathcal{D})) \otimes B_k & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \pi_*(\mathcal{L}(\mathcal{D})/\mathcal{L}) \otimes B_k & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

By the previous remarks diagram (2.9) is exact; in addition all the direct image sheaves appearing in it satisfy cohomology and base change by Corollary 2.5. So, setting $L_k := \mathcal{L}|_{kC}$ and $\mathcal{D}_k := \mathcal{D}|_{kC}$ it can be rewritten as:

(2.10)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathcal{W}_1 \otimes B_k) \cap (\mathcal{W}_2 \otimes B_k) & \longrightarrow & H^0(kC, L_k/L_k(-\mathcal{D}_k)) & \longrightarrow & H^1(kC, L_k(-\mathcal{D}_k)) & & \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H^0(kC, L_k(\mathcal{D}_k)) & \longrightarrow & H^0(kC, L_k(\mathcal{D}_k)/L_k(-\mathcal{D}_k)) & \longrightarrow & H^1(kC, L_k(-\mathcal{D}_k)) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & H^0(kC, L_k(\mathcal{D}_k)/L_k) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

The first row of (2.10) gives an identification $H^0(kC, L_k) \cong (\mathcal{W}_1 \otimes B_k) \cap (\mathcal{W}_2 \otimes B_k)$. Denote by V, resp. W_1, W_2, the stalks at $\bar{t}$ of $\mathcal{V}$, resp. $\mathcal{W}_1, \mathcal{W}_2$, so that $\mathcal{V} \otimes B_k = V \otimes B_k$, resp. $\mathcal{W}_1 \otimes B_k = W_1 \otimes B_k$, $\mathcal{W}_2 \otimes B_k = W_2 \otimes B_k$. Our claim now follows by Lemma 2.3 provided one extends the $\mathbb{K}$-bilinear form $\overline{Q} \colon \overline{V} \times \overline{V} \to \mathbb{K}$ to an A-bilinear form $Q \colon V \times V \to A$ such that W_1 and

W_2 are isotropic subspaces. This boils down to being able to take residues of rational sections of $\mathcal{L}^{\otimes 2}$ along the components of $\mathcal{D}$ and it can be done as in the proof of [Ha82, Theorem 1.10.(i)]. The surface X is Gorenstein, since Δ is smooth and π has Gorenstein fibers, and locally near C there is an isomorphism $\mathcal{L}^{\otimes 2} \cong \omega_\pi$. In turn, ω_π restricts to the sheaf relative differentials on the smooth locus of X.

2.3. **Proof of Corollary 1.2.** The statement is local, so we may assume that the torsion subsheaf $\mathcal{R}$ of $R^1\pi_*\mathcal{L}$ is supported at a single point $\bar{t} \in \Delta$. We write again $A := \mathcal{O}_{\Delta,\bar{t}}$ and $B_k := A/s^kA$ for $s \in A$ a local parameter. Since A is a DVR, there is a decomposition $\mathcal{R} = A/s^{r_1}A \oplus \cdots \oplus A/s^{r_m}A$, where $1 \le r_1 \le \cdots \le r_m$. For $j \in \mathbb{N}_{>0}$ we let m_j be the number of indices i such that $r_i \ge j$ (e.g., $m_1 = m$). The statement is equivalent to showing that all the m_j are even.

For $t \in \Delta$ denote by C_t the fiber of π over t and set $C := C_{\bar{t}}$, $L_k := \mathcal{L}|_{kC}$. Denote by q_0 the rank of $R^1\pi_*\mathcal{L}$; for $t \ne \bar{t} \in \Delta$, $q_0 = h^1(C_t, \mathcal{L}|_{C_t}) = h^0(C_t, \mathcal{L}|_{C_t})$, where the second equality follows by Riemann-Roch on C_t.

By Corollary 2.5, $h^1(kC, L_k) = kq_0 + m_1 + \cdots + m_k$. On the other hand, Riemann-Roch on kC gives $h^1(kC, L_k) = h^0(kC, L_k)$. By the constancy of the parity of theta characteristics in families ([Mu71, §1] and [Ha82, Theorem 1.10.(i)]), q_0 and $h^0(C, L)$ have the same parity, so m_1 is even. Now Theorem 1.1 implies that $m_1 + \cdots + m_k$ is even for every $k \ge 2$, hence all the m_j are even.

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